

Breathing Oscillations in Bose - Fermi Mixing Gases with Yb atoms in the Largely Prolate Deformed Traps

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We study the breathing oscillations in bose-fermi mixtures with Yb isotopes in the largely prolate deformed trap, which are realized by Kyoto group. We choose the three combinations of the Yb isotopes, ^{170}Yb - ^{171}Yb , ^{170}Yb - ^{173}Yb and ^{174}Yb - ^{173}Yb , whose boson-fermion interactions are weakly repulsive, strongly attractive and strongly repulsive. The collective oscillations in the deformed trap are calculated in the dynamical time-development approach, which is formulated with the time-dependent Gross-Pitaevskii and the Vlasov equations. We analyze the results in the time-development approach with the intrinsic oscillation modes of the deformed system, which are obtained using the scaling method, and show that the damping and forced-oscillation effects of the intrinsic modes give time-variation of oscillations, especially, in the fermion transverse mode.

I. INTRODUCTION

Over the last several years, there have been significant progresses in the production of ultracold gases, which realize the Bose-Einstein condensates (BEC) [1–4], two boson mixtures [5, 6], degenerate atomic Fermi gases [7], and Bose-Fermi (BF) mixing gases [5, 8–10]. In particular the BF mixtures attract physical interest as a typical example in which particles obeying different statistics are intermingled. Using the system we have a very big opportunity to get various new knowledge about many-body quantum systems because we can make a large variety of combinations of atomic species and control the atomic interactions using the Feshbach resonance [11].

Theoretical studies of the BF mixtures have been done for static properties [12–17], for the phase diagram and phase separation [18–21], for stability [22–24] and for collective excitations [14, 25, 26, 28–34].

Recently Kyoto university group has performed researches on the trapped atomic gases of the Yb isotopes: the BEC [35] and the Fermi-degeneracy [36]. They also succeeded in realizing

the BF mixtures of the isotopes. The Yb consists of many kinds of isotopes— five bosons ($^{168,170,172,174,176}\text{Yb}$) and two fermions ($^{171,173}\text{Yb}$), which give a variety of combinations in the BF mixtures. The scattering lengths for the boson-fermion interactions have been determined experimentally by the group [36–38], and the observation of the ground state properties and the collective oscillations of the BF mixtures is now under progressing.

In many characteristic properties, the spectrum of the collective excitations is an important diagnostic signal of these systems; they are common in many-particle systems and are often sensitive to the interaction and the structure of the ground and the excited states.

We have constructed a time-dependent dynamical approach with the time-dependent Gross-Pitaevskii (TDGP) and Vlasov equations, and have studied the monopole [31, 32] and dipole oscillations [33] of the BF mixtures. In these works, we have found that collective motions in the BF mixtures include various modes such as the boson and fermion intrinsic modes, the forced oscillation modes and so on. Then the intrinsic frequencies of these modes in our approach are different from those in the sum-rule [14, 28] and the scaling [39–42] approaches, which are approximations of the random phase approximation (RPA) [25, 26]. Furthermore the time-dependent behaviors in our approach are the same as those in RPA only in the early time, and show difference in latter time. As the boson-fermion interaction becomes stronger, the difference appears in earlier time. Thus the BF mixing gases show new dynamical properties different from the other finite many-body systems such as atomic nuclei.

We choose three combinations of the Yb isotopes, ^{170}Yb - ^{171}Yb , ^{170}Yb - ^{173}Yb and ^{174}Yb - ^{173}Yb , whose boson-fermion interactions are weakly repulsive, strongly attractive and strongly repulsive. In the previous paper [34], we have calculated the quadrupole oscillation of these mixtures in the spherical trap using the time-dependent dynamical approach, and then compared the results with those obtained in RPA. The RPA predicts the same intrinsic modes as the dynamical approach, but it describes the whole oscillations of the mixture as linear combination of the intrinsic modes. When the BF interaction is weak, and/or the amplitude is small, the linearly-combined oscillations can be good approximation, so that the RPA gives the results consistent with those in the dynamical approach. However, when the BF interaction is strong, or the amplitude is large, the oscillations of the BF mixtures obtained in the dynamical approach show the very different behaviors from the RPA; in the strongly-attractive BF interaction, the fermi gas overflows from the boson occupation region and makes large expansion, which leads to a monopole oscillation mode. In the case of the strongly-repulsive interaction, the fermion intrinsic-mode has rapid damping, so that the fermi gas comes to co-oscillate with the bose gas.

In actual experiments, the largely-deformed trap of axial-symmetry is used, and the oscillation behaviors may be different from those in the spherical trap. The monopole and quadrupole oscillations, which are coupled in axial-symmetric traps, are recombined into two modes oscillating along the symmetric axial and the transverse directions. Such behaviors are observed in the two-component fermi-gas oscillations [42], but no studies have been done for the BF mixture.

In this paper, we investigate the breathing oscillations of the BF mixtures in the largely deformed system of the prolate shape. Then we examine oscillation behaviors along the longitudinal and transverse directions separately, which are defined as the directions of the symmetry axis and perpendicular to it, respectively.

In the next section, we explain the dynamical model to calculate the time evolution of the system. In Sec. III, we examine the intrinsic collective modes and their coupling behaviors in the scaling method. In Sec. IV, we show the time-development calculations in the TDHF+Vlasov approach for the breathing oscillations in the BF mixture, and discuss their properties. The summary of the present work is given in Sec. V.

II. FORMALISM

A. Total Hamiltonian

In this work, we consider the mixture of a dilute boson and one-component-fermion gases at zero temperature in the trapping potential of the axial symmetry; the symmetry axis is chosen to be the z -axis. We assume zero-range interactions between atoms, and no fermion-fermion interactions. The hamiltonian is given by

$$\begin{aligned} \tilde{H} = \int d^3q \left[-\frac{\hbar^2}{2M_B} \tilde{\phi}^\dagger(\mathbf{q}) \nabla_q^2 \tilde{\phi}(\mathbf{q}) + \frac{1}{2} M_B \Omega_B^2 (\mathbf{q}_T^2 + \kappa_L^2 q_L^2) \tilde{\phi}^\dagger(\mathbf{q}) \tilde{\phi}(\mathbf{q}) \right. \\ + \frac{2\pi\hbar^2 a_{BB}}{M_B} \{ \tilde{\phi}^\dagger(\mathbf{q}) \tilde{\phi}(\mathbf{q}) \}^2 \\ - \frac{\hbar^2}{2M_f} \tilde{\psi}^\dagger(\mathbf{q}) \nabla_q^2 \tilde{\psi}(\mathbf{q}) + \frac{1}{2} M_f \Omega_F^2 (\mathbf{q}_T^2 + \kappa_L^2 q_L^2) \tilde{\psi}^\dagger(\mathbf{q}) \tilde{\psi}(\mathbf{q}) \\ \left. + \frac{2\pi\hbar^2 a_{BF} (M_B + M_F)}{M_B M_F} \tilde{\phi}^\dagger(\mathbf{q}) \tilde{\phi}(\mathbf{q}) \tilde{\psi}^\dagger(\mathbf{q}) \tilde{\psi}(\mathbf{q}) \right], \end{aligned} \quad (1)$$

where $\tilde{\phi}$ and $\tilde{\psi}$ are boson and fermion fields, M_B and M_F are the boson and fermion masses, Ω_B and Ω_F are the transverse trapped frequencies of the boson and the fermion, a_{BB} and a_{BF} are the boson-boson and boson-fermion s -wave scattering lengths. The $\mathbf{q} \equiv (\mathbf{q}_T, q_L)$ is the positional coordinate, and κ_L is the trapping-potential frequency ratio of the longitudinal to the transverse directions.

With the boson harmonic oscillator length for the transverse direction $R_B = (\hbar/M_B\Omega_B)^{1/2}$, we introduce the dimensionless coordinate $\mathbf{r} \equiv (r_1, r_2, r_3) \equiv (\mathbf{r}_T, z) \equiv \mathbf{q}/R_B$, and the scaled boson and fermion fields, $\phi = R_B^{-1/3}\tilde{\phi}$ and $\psi = R_B^{-1/3}\tilde{\psi}$. Using the variable and fields, the scaled hamiltonian $H \equiv \tilde{H}/\hbar\Omega_B$ becomes

$$H = \int d^3r \left[-\frac{1}{2}\phi^\dagger(\mathbf{r})\nabla_r^2\phi(\mathbf{r}) + \frac{1}{2}(\mathbf{r}_T^2 + \kappa_L^2 z^2)\phi^\dagger(\mathbf{r})\phi(\mathbf{r}) + \frac{g_{BB}}{2}\{\phi^\dagger(\mathbf{r})\phi(\mathbf{r})\}^2 \right. \\ \left. -\frac{1}{2m_f}\psi^\dagger(\mathbf{r})\nabla_r^2\psi(\mathbf{r}) + \frac{1}{2}m_f\omega_f^2(\mathbf{r}_T^2 + \kappa_L^2 z^2)\psi^\dagger(\mathbf{r})\psi(\mathbf{r}) \right. \\ \left. + h_{BF}\phi^\dagger(\mathbf{r})\phi(\mathbf{r})\psi^\dagger(\mathbf{r})\psi(\mathbf{r}) \right], \quad (2)$$

where m_f and ω_f are the fermion-boson ratio of the mass and the trapping-potential frequency, $m_f \equiv M_F/M_B$ and $\omega_f \equiv \Omega_F/\Omega_B$, respectively, and the dimensionless coupling constants of the boson-boson and boson-fermion interactions are defined by $g_{BB} \equiv 8\pi\hbar a_{BB}R_B^{-1}$ and $h_{BF} \equiv 4\pi\hbar a_{BF}m_f(1+m_f)^{-1}R_B^{-1}$.

At zero-temperature, the total wave function of the BF-mixture including N_b bosons and N_f fermions is written by

$$\Phi(\tau) = \left\{ \prod_{i=1}^{N_b} \phi_c(\mathbf{r}_i) \right\} \Psi_f[\psi_n], \quad (3)$$

where ϕ_c is a wave function of the condensed boson and Ψ_f is a Slater determinant of fermion single-particle wave functions ψ_n .

The total energy is given by

$$E_T = \int d^3r \left[-\frac{1}{2}\nabla_r\phi_c^\dagger(\mathbf{r})\nabla_r\phi_c(\mathbf{r}) + \frac{1}{2}(\mathbf{r}_T^2 + \kappa_L^2 z^2)\phi_c^\dagger(\mathbf{r})\phi_c(\mathbf{r}) + \frac{g_{BB}}{2}\{\phi_c^\dagger(\mathbf{r})\phi_c(\mathbf{r})\}^2 \right. \\ \left. + \frac{1}{2m_f} \sum_n \nabla_r\psi_n^\dagger(\mathbf{r})\nabla_r\psi_n(\mathbf{r}) + \frac{1}{2}m_f\omega_f^2(\mathbf{r}_T^2 + \kappa_L^2 z^2) \sum_n \psi_n^\dagger(\mathbf{r})\psi_n(\mathbf{r}) \right. \\ \left. + h_{BF}\phi_c^\dagger(\mathbf{r})\phi_c(\mathbf{r}) \sum_n \psi_n^\dagger(\mathbf{r})\psi_n(\mathbf{r}) \right]. \quad (4)$$

B. Time Evolution Equation

The time evolution of the wave functions are obtained from the variational condition:

$$\delta \int d\tau \langle \Phi(\tau) | \left(i \frac{\partial}{\partial \tau} - H \right) | \Phi(\tau) \rangle = 0. \quad (5)$$

From the condition, the coupled TDGP and TDHF equations are derived:

$$i \frac{\partial}{\partial \tau} \phi_c(\mathbf{r}, \tau) = \left\{ -\frac{1}{2}\nabla_r^2 + U_B(\mathbf{r}) \right\} \phi_c(\mathbf{r}, \tau), \quad (6)$$

$$i \frac{\partial}{\partial \tau} \psi_n(\mathbf{r}, \tau) = \left\{ -\frac{1}{2m_f}\nabla_r^2 + U_F(\mathbf{r}) \right\} \psi_n(\mathbf{r}, \tau). \quad (7)$$

The effective potentials U_B and U_F are defined by

$$U_B(\mathbf{r}) = \frac{1}{2}(\mathbf{r}_T^2 + \kappa_L^2 z^2) + g_{BB}\rho_B(\mathbf{r}) + h_{BF}\rho_F(\mathbf{r}), \quad (8)$$

$$U_F(\mathbf{r}) = \frac{1}{2}m_f\omega_f^2(\mathbf{r}_T^2 + \kappa_L^2 z^2) + h_{BF}\rho_B(\mathbf{r}), \quad (9)$$

where ρ_B and ρ_F are the boson/fermion densities:

$$\rho_B(\mathbf{r}) = N_b |\phi_c(\mathbf{r})|^2, \quad (10)$$

$$\rho_F(\mathbf{r}) = \sum_n^{occ} |\psi_n(\mathbf{r})|^2. \quad (11)$$

The number of fermion states necessary for calculation are usually too large to solve the above TDHF equations directly, so that we use the semi-classical approach. In the semi-classical limit ($\hbar \rightarrow 0$), the TDHF equation is equivalent to the Vlasov equation [43]:

$$\frac{d}{d\tau}f(\mathbf{r}, \mathbf{p}; \tau) = \left\{ \frac{\partial}{\partial \tau} + \frac{\mathbf{p}}{m_f} \cdot \nabla_r - [\nabla_r U_F(\mathbf{r})] \cdot \nabla_p \right\} f(\mathbf{r}, \mathbf{p}; \tau) = 0, \quad (12)$$

where $f(\mathbf{r}, \mathbf{p}; \tau)$ is the fermion phase-space distribution function:

$$f(\mathbf{r}, \mathbf{p}, \tau) = \int d^3u \langle \Phi | \psi \left(\mathbf{r} + \frac{1}{2}\mathbf{u}, \tau \right) \psi^\dagger \left(\mathbf{r} - \frac{1}{2}\mathbf{u}, \tau \right) | \Phi \rangle e^{-i\mathbf{p}\mathbf{u}}. \quad (13)$$

The time-evolution of the system is obtained by solving Eqs. (6) and (12).

C. Ground State

In the ground state, the condensed-boson wave function ϕ_c is obtained from the solution of the Gross-Pitaevskii (GP) and the Hartree-Fock (HF) equations:

$$\left\{ -\frac{1}{2}\nabla_r^2 + U_B(\mathbf{r}) \right\} \phi_c(\mathbf{r}, \tau) = \mu_b \phi_c(\mathbf{r}), \quad (14)$$

$$\left\{ -\frac{1}{2m_f}\nabla_r^2 + U_F(\mathbf{r}) \right\} \psi_n(\mathbf{r}, \tau) = \varepsilon_n \psi_n(\mathbf{r}, \tau), \quad (15)$$

where μ_b is the boson chemical potential, and ε_n is the fermion single-particle energy of the n -th state.

In the semi-classical limit, the fermion phase-space distribution function in the HF approximation becomes the same one in the Thomas-Fermi (TF) approximation:

$$f(\mathbf{r}, \mathbf{p}) = \theta[\mu_f - \varepsilon(\mathbf{r}, \mathbf{p})], \quad (16)$$

where μ_f is the fermion chemical potential, and the TF energy $\varepsilon(\mathbf{r}, \mathbf{p})$ is defined by

$$\varepsilon(\mathbf{r}, \mathbf{p}) = \frac{1}{2m_f}\mathbf{p}^2 + U_F(\mathbf{r}), \quad (17)$$

In this TF approximation, the fermion density ρ_F is obtained from the TF equation:

$$\frac{1}{2m_f} \{6\pi^2 \rho_F(\mathbf{r})\}^{2/3} + \frac{1}{2} m_f \omega_f^2 (\mathbf{r}_T^2 + \kappa_L^2 z^2) + h_{BF} \rho_B(\mathbf{r}) = \mu_f. \quad (18)$$

When $N_b \ll 1$, in good approximation, the boson density ρ_B is obtained from the TF equation:

$$\frac{1}{2} (\mathbf{r}_T^2 + \kappa_L^2 z^2) + g_{BB} \rho_B(\mathbf{r}) + h_{BF} \rho_F(\mathbf{r}) = \mu_b. \quad (19)$$

In actual calculation, we use the boson and fermion densities obtained in the TF equations (18) and Eq. (19) as the initial inputs of the iteration; using the TF densities in Eq. (14), we obtain the boson wave function ϕ_c , which gives the next-step ρ_B in Eq. (10). We iterate solving ϕ_c and searching the fermi energy μ_f in Eq. (18) to give the correct fermion number,

III. BREATHING OSCILLATIONS IN THE SCALING METHOD

The deformed system of BF mixtures generally has the coupled oscillation of the monopole and quadrupole modes, which are the breathing oscillation.

The mode-structure of breathing oscillation in the deformed system has been studied, for example, in two-component fermi gas [42], but it is not clear for the deformed BF mixtures. Thus, before presenting the calculation of the time-evolution, we should clarify the mode-structure of the breathing oscillations in the deformed BF mixture using the scaling method as in Ref. [44]; it is a macroscopic approximation consistent with the energy-weighted sum rules [45].

A. Scaled Wave Functions

In the scaling method, the scaled wave functions are introduced on the basis of the ground-state one-body wave functions:

$$\phi_\lambda(\mathbf{r}, \tau) = e^{i\xi_B(\mathbf{r}, \tau)} e^{\lambda_{T,B}(\tau) + \frac{1}{2}\lambda_{L,B}(\tau)} \phi_c(e^{\lambda_{T,B}(\tau)} \mathbf{r}_T; e^{\lambda_{L,B}(\tau)} z), \quad (20)$$

$$\psi_{\lambda,n}(\mathbf{r}, \tau) = e^{im_f \xi_F(\mathbf{r}, \tau)} e^{\lambda_{T,F}(\tau) + \frac{1}{2}\lambda_{L,F}(\tau)} \psi_n(e^{\lambda_{T,F}(\tau)} \mathbf{r}_T; e^{\lambda_{L,F}(\tau)} z), \quad (21)$$

where

$$\xi_a(\mathbf{r}, \tau) = \frac{1}{2} \left[\dot{\lambda}_{T,a}(\tau) \mathbf{r}_T^2 + \dot{\lambda}_{L,a}(\tau) z^2 \right], \quad (a = B, F). \quad (22)$$

The factor $\exp(i\xi_a)$ is the Gallilei-transformation factor, which is necessary for the scaled wave functions to satisfy the continuum equation.

The variables, $\lambda_{T,B}$, $\lambda_{L,B}$, $\lambda_{T,F}$, $\lambda_{L,F}$, are the collective coordinates describing the boson longitudinal breathing (BLB), boson transverse breathing (BTB), fermion longitudinal breathing (FLB)

and fermion transverse breathing (FTB) oscillation modes, and $\dot{\lambda}$'s are the time-derivatives of them. In the BF mixtures of $g_{BB} = h_{BF} = 0$ (no interactions), the matrices B and C become diagonal, so that the four breathing modes are completely decoupled.

Substituting the wave-functions (20,21) into the total energy functional (4), we obtain the total energy:

$$\begin{aligned}
E_T = & \frac{1}{2} \int d^3r \left\{ (\nabla_r \xi_B)^2 \rho_B + e^{2\lambda_{T,B}} (T_{B,1} + T_{B,2}) + e^{2\lambda_{L,B}} T_{B,3} \right\} \\
& + \frac{1}{2} \int d^3r \left\{ e^{-2\lambda_{T,B}} (r_1^2 + r_2^2) + e^{-2\lambda_{L,B}} \kappa_L^2 z^2 \right\} \rho_B(\mathbf{r}) \\
& + \frac{g_{BB}}{2} e^{2\lambda_{T,B} + \lambda_{L,B}} \int d^3r \rho_B^2(\mathbf{r}), \\
& + \frac{1}{2m_f} \int d^3r \left\{ m_f^2 (\nabla_r \xi_F)^2 \rho_F + e^{2\lambda_{T,F}} (T_{F,1} + T_{F,2}) + e^{2\lambda_{L,F}} T_{F,3} \right\} \\
& + \frac{1}{2} m_f \omega_f^2 \int d^3r \left\{ e^{-2\lambda_{F,s}} (r_1^2 + r_2^2) + e^{-2\lambda_{F,s}} \kappa_L^2 z^2 \right\} \rho_F(\mathbf{r}), \\
& + h_{BF} e^{2\lambda_{T,B} + \lambda_{L,B} + 2\lambda_{T,F} + \lambda_{L,F}} \int d^3r \rho_B(e^{\lambda_{T,B}} \mathbf{r}_T; e^{\lambda_{L,B}} z) \rho_F(e^{\lambda_{T,F}} \mathbf{r}_T; e^{\lambda_{L,F}} z), \quad (23)
\end{aligned}$$

where

$$T_B = \int d^3r |\nabla_r \phi(\mathbf{r})|^2, \quad (24)$$

$$T_{F,i} = \sum_n \int d^3r \left| \frac{\partial}{\partial r_i} \psi_n(\mathbf{r}) \right|, \quad (i = 1, 2, 3). \quad (25)$$

In order to evaluate the ground state of the system we use the Thomas-Fermi (TF) approximation. where the momentum distribution becomes spherically-symmetric, the T_B and $T_{F,i}$ become

$$T_B(\mathbf{r}) = 0, \quad (26)$$

$$T_{F,1}(\mathbf{r}) = T_{F,2}(\mathbf{r}) = T_{F,3}(\mathbf{r}) = \frac{1}{5} (6\pi^2)^{\frac{2}{3}} [\rho_F(\mathbf{r})]^{\frac{5}{3}}. \quad (27)$$

In order to simplify the analytic equations, we use the dimensionless variables:

$$h = \frac{1}{m_f \omega_f^2} \frac{h_{BF}}{g_{BB}}, \quad \mathbf{x} = \frac{m_f^4 \omega_f^5 g_{BB}}{3\pi^2} (r_1, r_2, \kappa_L z), \quad (28)$$

$$n_B = \frac{2m_f^8 \omega_f^{10} g_{BB}^3}{9\pi^4} \rho_B, \quad n_F = \frac{2m_f^9 \omega_f^{12} g_{BB}^3}{9\pi^4} \rho_F, \quad (29)$$

$$e_B = \frac{2m_f^8 \omega_f^{10} g_{BB}^2}{9\pi^4} \mu_B, \quad e_F = \frac{2m_f^7 \omega_f^8 g_{BB}^2}{9\pi^4} \mu_F, \quad (30)$$

we obtain the scaled total energy, which is also dimensionless:

$$\begin{aligned}
\tilde{E}_T^{(0)} &= \frac{2\kappa_L m_f^{28} \omega_f^{35} g_{BB}^8}{3^7 \pi^{14}} E_T(\lambda = 0, \dot{\lambda} = 0) \\
&= \int d^3x \left\{ x^2 n_B + \frac{1}{2} n_B^2 + \frac{3}{5} n_F^{\frac{5}{3}} + x^2 n_F + h n_B n_F \right\}, \quad (31)
\end{aligned}$$

where $x^2 = |\mathbf{x}|^2$. The variations of the total energy, $\delta\tilde{E}^{(0)}/\delta n_B = 0$ and $\delta\tilde{E}^{(0)}/\delta n_F = 0$, give the TF equations:

$$n_B + \hbar n_F = e_B - x^2, \quad (32)$$

$$n_F^{\frac{2}{3}} + \hbar n_F = e_F - x^2. \quad (33)$$

From the equation of the second variation:

$$\frac{\delta^2 \tilde{E}^{(0)}}{\delta n_B^2} \frac{\delta^2 \tilde{E}^{(0)}}{\delta n_F^2} - \left(\frac{\delta^2 \tilde{E}^{(0)}}{\delta n_B \delta n_F} \right)^2 = \frac{2}{3} n_F^{-\frac{1}{3}} - \hbar^2 > 0. \quad (34)$$

we obtain the stability condition of the TF solution:

$$n_F < \frac{8}{27\hbar^6}. \quad (35)$$

B. Collective Oscillations

Now we study the collective oscillation in the scaling method. The total energy for collective oscillations (4) is written by

$$\begin{aligned} \tilde{E}_T = & \frac{1}{2} \tilde{X}_B \left\{ \frac{2e^{-2\lambda_{T,B}}}{3} \dot{\lambda}_{T,B}^2 + \frac{e^{-2\lambda_{L,B}}}{3\kappa_L^2} \dot{\lambda}_{L,B}^2 \right\} \\ & + \frac{2e^{-2\lambda_{T,B}} + e^{-2\lambda_{L,B}}}{3} \tilde{X}_B + e^{2\lambda_{T,B} + \lambda_{L,B}} V_{bb} \\ & + \frac{1}{2} \tilde{X}_F \omega_f^2 \left\{ \frac{2e^{-2\lambda_{T,F}}}{3} \dot{\lambda}_{T,F}^2 + \frac{e^{-2\lambda_{L,F}}}{3\kappa_L^2} \dot{\lambda}_{L,F}^2 \right\} \\ & + \frac{2e^{2\lambda_{T,F}} + e^{2\lambda_{L,F}}}{3} \tilde{T}_F + \frac{2e^{-2\lambda_{T,F}} + e^{-2\lambda_{L,F}}}{3} \tilde{X}_F + \tilde{V}_{bf} \end{aligned} \quad (36)$$

where

$$\tilde{T}_F = \frac{3}{5} \int d^3x n_F^{\frac{5}{3}}(\mathbf{x}) \quad (37)$$

$$\tilde{X}_{B,F} = \int d^3x x^2 n_{B,F}(\mathbf{x}) \quad (38)$$

$$\tilde{V}_{bb} = \int d^3x n_B^2(\mathbf{x}) \quad (39)$$

$$\begin{aligned} \tilde{V}_{bf} = & e^{2\lambda_{T,B} + \lambda_{L,B} + 2\lambda_{T,F} + \lambda_{L,F}} \\ & \times \hbar \int d^3r n_B(e^{\lambda_{T,B}} \mathbf{x}_T, e^{\lambda_{L,B}} x_3) n_F(e^{\lambda_{T,F}} \mathbf{x}_T, e^{\lambda_{L,F}} x_3) \end{aligned} \quad (40)$$

Using the vector notation for the collective variables, $\mathbf{t}\boldsymbol{\lambda} = (\lambda_{T,B}, \lambda_{L,B}, \lambda_{T,F}, \lambda_{L,F})$, and expanding the total energy to the order of $O(\lambda^2)$, we obtain the oscillation energy of the system:

$$\Delta\tilde{E}_T \equiv \tilde{E}_T - \tilde{E}_T^{(0)} \approx \frac{1}{2} \mathbf{t} \dot{\boldsymbol{\lambda}} B \dot{\boldsymbol{\lambda}} + \frac{1}{2} \mathbf{t} \boldsymbol{\lambda} C \boldsymbol{\lambda}. \quad (41)$$

The matrices B and C are defined by where

$$B = \begin{pmatrix} \frac{4\tilde{X}_B}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3\kappa_L^2}\tilde{X}_B & 0 & 0 \\ 0 & 0 & \frac{4}{3\omega_f^2}\tilde{X}_F & 0 \\ 0 & 0 & 0 & \frac{2}{3\kappa_L^2\omega_f^2}\tilde{X}_F \end{pmatrix} \quad (42)$$

and

$$C = \begin{pmatrix} \frac{8}{3}\tilde{X}_B + 4V_{bb} & 2V_{bb} & 0 & 0 \\ 2V_{bb} & \frac{4}{3}\tilde{X}_B + V_{bb} & 0 & 0 \\ 0 & 0 & \frac{8}{3}(\tilde{T}_F + \tilde{X}_F) & 0 \\ 0 & 0 & 0 & \frac{4}{3}(\tilde{T}_F + \tilde{X}_F) \end{pmatrix} + \begin{pmatrix} -\frac{4}{3}V_1 - \frac{8}{15}V_3 & -\frac{2}{3}V_1 - \frac{2}{15}V_3 & \frac{8}{15}V_3 & \frac{2}{15}V_3 \\ -\frac{2}{3}V_1 - \frac{2}{15}V_3 & -\frac{1}{3}V_1 - \frac{1}{5}V_3 & \frac{2}{15}V_3 & \frac{1}{5}V_3 \\ \frac{8}{15}V_3 & \frac{2}{15}V_3 & -\frac{4}{3}V_2 - \frac{8}{15}V_3 & -\frac{2}{3}V_2 - \frac{2}{15}V_3 \\ \frac{2}{15}V_3 & \frac{1}{5}V_3 & -\frac{2}{3}V_2 - \frac{2}{15}V_3 & -\frac{1}{3}V_2 - \frac{1}{5}V_3 \end{pmatrix}, \quad (43)$$

where

$$V_1 = h \int d^3x x n_B \frac{\partial n_F}{\partial x}, \quad (44)$$

$$V_2 = h \int d^3x x n_F \frac{\partial n_B}{\partial x}, \quad (45)$$

$$V_3 = h \int d^3x x^2 \frac{\partial n_B}{\partial x} \frac{\partial n_F}{\partial x}. \quad (46)$$

The stability conditions of the breathing-oscillation states are obtained from $\partial \tilde{E}_T / \lambda_\alpha = 0$; for (41), they become

$$-2\tilde{X}_B + 3V_{bb} - V_1 = 0 \quad (47)$$

$$2\tilde{T}_F - 2\tilde{X}_F - V_2 = 0. \quad (48)$$

Eq. (41) shows that the classical equation of motion for $\boldsymbol{\lambda}$ is harmonic, so that the oscillation frequency ω and the corresponding amplitude $\boldsymbol{\lambda}$ are obtained from the characteristic equation:

$$(B\omega^2 - C) \boldsymbol{\lambda} = 0. \quad (49)$$

In the case of no interactions ($g_{BB} = h_{BF} = 0$), the matrices B and C become diagonal; thus the four modes of BLB, BTB, FLB, FTB are completely decoupled.

C. Boson- and Fermion-Intrinsic Oscillations

The collective oscillation obtained from (49) describes the mode where bosons and fermions oscillate with the same eigen-frequency ω (co-oscillating mode). In RPA or the time-evolution approach, the BF mixtures sometimes oscillate in the mode where the bosons and the fermions oscillate with the different frequencies. Thus, in the opposite case for co-oscillating modes, we should obtain the boson- and fermion-intrinsic oscillation modes where only the bosons or the fermions in the BF mixture oscillate collectively, especially in the case of large deformation ($\kappa_L \ll 1$).

1. boson-intrinsic modes:

The boson-intrinsic modes are obtained by putting the constraint $\lambda_{a,F} = 0$ ($a = T, L$) in (41); the characteristic equation become

$$\frac{2\tilde{X}_B}{3} \begin{pmatrix} 8 - \frac{4}{5}v_B - 2\omega^2 & 2 - \frac{1}{5}v_B \\ 2 - \frac{1}{5}v_B & 3 - \frac{3}{10}v_B - \frac{\omega^2}{\kappa_L^2} \end{pmatrix} \begin{pmatrix} \lambda_{T,B} \\ \lambda_{L,B} \end{pmatrix} = 0 \quad (50)$$

where $v_B = V_3/\tilde{X}_B$. The eigen-frequencies $\omega = \omega_B$ are obtained by

$$\omega_B^2 = \left(1 - \frac{1}{10}v_B\right) \left\{ 2 + \frac{3}{2}\kappa_L^2 \pm \sqrt{4 - 4\kappa_L^2 + \frac{9}{4}\kappa_L^4} \right\}. \quad (51)$$

In the spherically-symmetric case ($\kappa_L = 1$), the ω_B in (51) become $\omega_B^2 = 2(1 - v_B/10)$ and $5(1 - v_B/10)$; the corresponding amplitudes are $(\lambda_{T,B}, \lambda_{L,B}) = (1, -2)$ (quadrupole oscillation) and $(\lambda_{T,B}, \lambda_{L,B}) = (1, 1)$ (monopole oscillation) each other.

In the limit of large deformation ($\kappa_L \ll 1$), the boson-intrinsic frequencies ω_B become

$$\omega_B^2 \left(1 - \frac{1}{10}v_B\right)^{-1} \approx 4 \quad \text{and} \quad \frac{5}{2}\kappa_L^2. \quad (52)$$

For $\omega_B^2 \approx 4(1 - v_B/10)$, the eigenvector is given by

$$\begin{pmatrix} \lambda_{T,B} \\ \lambda_{L,B} \end{pmatrix} \approx \begin{pmatrix} 1 \\ \kappa_L^2 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (53)$$

It is just the transverse oscillation (BTB) mode. The eigenvector of the mode with $\omega_B^2 \approx \frac{5}{2}\kappa_L^2(1 - v_B/10)$ is

$$\begin{pmatrix} \lambda_{T,B} \\ \lambda_{L,B} \end{pmatrix} \approx \begin{pmatrix} -1 + \frac{10}{17}\kappa_L^2 \\ 4 + \frac{5}{34}\kappa_L^2 \end{pmatrix} \approx \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \quad (54)$$

where the bosons oscillates both the longitudinal and transverse directions in out-of-phase. This mode is similar to the quadrupole oscillation, but the transverse-to-longitudinal amplitude ratio is different (1 : 4). According to Ref. [46], we call it the boson axial-breathing (BAB) mode.

2. *fermion-intrinsic modes:*

Putting $\lambda_{a,B} = 0$ ($a = T, L$) in (41), we obtain the characteristic equation for the fermion-intrinsic mode:

$$\frac{2\tilde{X}_F}{3} \begin{pmatrix} 8 - \frac{4}{5}v_3 - 2\frac{\omega^2}{\omega_f^2} & -v_2 - \frac{1}{5}v_3 \\ -v_2 - \frac{1}{5}v_3 & 4 + \frac{1}{2}v_2 - \frac{3}{10}v_3 - \frac{\omega^2}{\kappa_L^2\omega_f^2} \end{pmatrix} \begin{pmatrix} \lambda_{T,F} \\ \lambda_{L,F} \end{pmatrix} = 0, \quad (55)$$

where $v_2 = V_2/\tilde{X}_F$ and $v_3 = V_3/\tilde{X}_F$. The eigen-frequencies $\omega = \omega_F$ are given by

$$\begin{aligned} \omega_F^2/\omega_f^2 = & 2\left(1 - \frac{v_3}{10}\right) + 2\left(1 + \frac{v_2}{8} - \frac{3v_3}{40}\right)\kappa_L^2 \\ & \pm 2\sqrt{\left[\left(1 - \frac{v_3}{10}\right) - \left(1 + \frac{v_2}{8} - \frac{3v_3}{40}\right)\kappa_L^2\right]^2 + \frac{1}{8}\left(v_2 + \frac{v_3}{5}\right)^2\kappa_L^2}, \end{aligned} \quad (56)$$

In the spherically-symmetric case ($\kappa_L = 1$), they become $\omega_F^2/\omega_f^2 = 4 + v_2 - v_3/5$ and $\omega_F^2/\omega_f^2 = 4 - v_2/2 - v_3/2$; the corresponding amplitudes are given by $(\lambda_{T,F}, \lambda_{L,F}) = (1, -2)$ (quadrupole oscillation) and $(\lambda_{T,F}, \lambda_{L,F}) = (1, 1)$ (monopole oscillation) each other.

In the limit of $\kappa_L \ll 1$, the ω_F^2 becomes

$$\omega_B^2/\omega_f^2 \approx \left(1 - \frac{v_3}{10}\right) \quad \text{and} \quad \left[4\left(1 + \frac{v_2}{8} - \frac{3v_3}{40}\right) - \frac{1}{8}\frac{\left(v_2 + \frac{v_3}{5}\right)^2}{\left(1 - \frac{v_3}{10}\right)}\right]\kappa_L^2. \quad (57)$$

The mode with $\omega^2 \approx 4(1 - v_3/10)$ is the transverse oscillation mode:

$$\begin{pmatrix} \lambda_{T,F} \\ \lambda_{L,F} \end{pmatrix} \approx \begin{pmatrix} 1 \\ \frac{1}{4(1 - \frac{v_3}{10})}\kappa_L^2 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (58)$$

We call it the fermion transverse breathing (FTB) mode, for convenience. The eigenvector of another mode is given by

$$\begin{pmatrix} \lambda_{T,F} \\ \lambda_{L,F} \end{pmatrix} \approx \begin{pmatrix} \frac{v_2 + \frac{v_3}{5}}{8(1 - \frac{v_3}{10})} \\ 1 \end{pmatrix}, \quad (59)$$

where the FLB and FTB modes are coupled through the boson-fermion interactions. We call this new type of oscillation the fermion axial breathing (FAB) mode.

In Fig. 1, we show the frequencies of the monopole (a) and quadrupole (b) oscillations in the spherically-symmetric case. The frequencies of the quadrupole oscillation have no level crossing, and co-oscillating collective modes (dotted line) are almost the same with the boson-/fermion-intrinsic modes. In the monopole oscillation, the frequencies of the boson and fermion modes have crossing points at $h_{BF}/g_{BB} \sim -1.5$ and 2. In the region of $-1.5 \lesssim h_{BF}/g_{BB} \lesssim 2$, the co-oscillating collective oscillation almost agrees with the intrinsic modes, and, in the region,

$h_{BF}/g_{BB} \lesssim -1.5$ and $2 \lesssim h_{BF}/g_{BB}$, the co-oscillating mode is found to be different from the boson-/fermion-intrinsic modes.

In the same figure, we plot the boson- and fermion-intrinsic frequencies in the TDGP+Vlasov approach (the open squares and the full circles) [31, 34]. We find that the scaling method well-reproduces the results in the TDGP+Vlasov approach for the boson-fermion attractive interaction, but gives the different results when $h_{BF}/g_{BB} \gtrsim 0$ (monopole oscillation) and $h_{BF}/g_{BB} \gtrsim 1$ (quadrupole oscillation). The similar behavior is also found in the case of the dipole oscillation [33]; We have explained on this discrepancy in the previous papers [33, 34].

IV. RESULTS AND DISCUSSIONS

In this section, we show the calculations on the breathing oscillation mode of the deformed BF mixtures calculated in the time-development approach.

Kyoto group controls the interactions by changing the combinations of the Yb-isotopes. We search combinations which are available in the present calculation, and choose the three kinds of combinations ^{170}Yb - ^{171}Yb , ^{170}Yb - ^{173}Yb and ^{174}Yb - ^{173}Yb , where the boson-fermion interaction is weakly repulsive, strongly attractive and strongly repulsive, respectively. The coupling constants are shown in Table I.

We deal with the Yb-Yb system, where the number of the bosons and the fermions are $N_b = 10000$ and $N_f = 1000$, respectively. The parameters for the trapping potential is $\Omega_B = 2\pi \times 300$ (Hz), $\kappa_L = 1/6$ and $\omega_f = 1$: the same trap in the actual experiments. Furthermore, we use the same mass for all Yb-isotopes ($m_f = 1$); the mass differences can be safely neglected for Yb isotopes..

A. Ground States

In Fig. 2, we show the boson and fermion density distributions of the ground states for ^{170}Yb - ^{171}Yb (upper panels), ^{170}Yb - ^{173}Yb (middle panels) and ^{174}Yb - ^{173}Yb (bottom panels). The boson densities are center-peaked in all cases. In contrast, the fermion densities have a center-peaked shape largely distributed in the boson-distributed regions in attractive BF interactions $h_{BF}/g_{BB} < 0$ (^{170}Yb - ^{171}Yb and ^{170}Yb - ^{173}Yb in Figs. 2a and 2b), and have a surface-peaked shape when $h_{BF}/g_{BB} > 1$ (^{174}Yb - ^{173}Yb in Fig. 2c). This BF interaction dependence of the fermion density distribution can be easily explained in the TF approximation, where the ground-state density is determined not by h_{BF} and g_{BB} but through the ratio h_{BF}/g_{BB} .

B. Description of Oscillations

In order to solve the Vlasov equation (12) numerically, we use the test-particle method [47]; using the test-particle coordinates and momenta \mathbf{r}_i and \mathbf{p}_i , the fermion phase-space distribution function is written as

$$f(\mathbf{r}, \mathbf{p}, \tau) = \frac{(2\pi)^3}{\tilde{N}_T} \sum_{i=1}^{\tilde{N}_T N_f} \delta\{\mathbf{r} - \mathbf{r}_i(\tau)\} \delta\{\mathbf{p} - \mathbf{p}_i(\tau)\}. \quad (60)$$

where \tilde{N}_T is the number of test-particles per fermion.

Substituting Eq. (60) into Eq. (12), we can obtain the equations of motion for test-particles:

$$\frac{d}{d\tau} \mathbf{r}_i(\tau) = \frac{\mathbf{p}_i}{m_f}, \quad (61)$$

$$\frac{d}{d\tau} \mathbf{p}_i(\tau) = -\nabla_{\mathbf{r}} U_F(\mathbf{r}_i). \quad (62)$$

As the initial conditions of time-development at $\tau = 0$, we use the boosted condensed-boson wave function and the fermion test-particle coordinates:

$$\phi_c(\mathbf{r}, \tau = 0) = \exp \left\{ \frac{i}{2} (b_T r_T^2 + b_L z^2) \right\} \phi_c^{(g)}(\mathbf{r}), \quad (63)$$

$$\mathbf{p}_T(i) = \mathbf{p}_T^{(g)}(i) + m_f \omega_f c_T \mathbf{r}_T(i), \quad p_z(i) = p_z^{(g)}(i) - 2m_f \kappa_L \omega_f c_L z(i) \quad (64)$$

where b_T , b_L , c_T and c_L are the boost parameters, and the superscript (g) represents the ground state. Then, the initial boson/fermion current densities $\mathbf{j}_{B,F}(\mathbf{r})$ become

$$\mathbf{j}_B(\mathbf{r}, \tau = 0) = (b_T \mathbf{r}_T + b_L z \hat{z}) \rho_B^{(g)}(\mathbf{r}), \quad (65)$$

$$\mathbf{j}_F(\mathbf{r}, \tau = 0) = \omega_f (c_T \mathbf{r}_T + c_L z \hat{z}) \rho_F^{(g)}(\mathbf{r}). \quad (66)$$

Solving Eqs. (6), (61) and (62) with the above initial states, we obtain the time-development of the condensed boson wave function and the fermion phase space distribution function.

In order to examine the aspects of the breathing oscillation, we introduce the quantities:

$$\Delta x_L(\tau; s) = \frac{R_L(\tau; s)}{R_L^{(g)}(s)} - 1, \quad (67)$$

$$\Delta x_T(\tau; s) = \frac{R_T(\tau; s)}{R_T^{(g)}(s)} - 1, \quad (68)$$

where $s = B, F$, and $R_{L(T)}(B)$ and $R_{L(T)}$ are the root-mean-square radii of the bose and fermi density-distributions along longitudinal (transverse) directions. The quantities Δx represent the change of distributions in longitudinal and transverse directions.

In order to inspect the oscillation modes, we use the strength functions defined by the Fourier transform of $\Delta x_{L,T}(s)$ ($s = B, F$):

$$S_{L,T}(\omega; s) = \int_{t_i}^{t_f} d\tau \Delta x_{L,T}(\tau; s) \sin \omega \tau \quad (69)$$

$$C_{L,T}(\omega; s) = \int_{t_i}^{t_f} d\tau \Delta x_{L,T}(\tau; s)(\tau) \cos \omega \tau. \quad (70)$$

In actual calculation, we take $0 < \tau < 200$ for the integration interval unless otherwise noted. The strength functions $S_{T,L}$ and $C_{T,L}$ exhibit the time-odd and time-even components of $\Delta x_{L,T}(B, F; \tau)$, respectively. In the linear-response approximation, only the $S_{T,L}$ has strength for the present initial conditions: $\Delta x_{L,T} = 0$ and $d\Delta x_{L,T}/d\tau \neq 0$ at $\tau = 0$.

C. Longitudinal Breathing Oscillations

First, we present numerical calculations in the TDGP+Vlasov approach with the longitudinally-deformed initial condition, $b_T = c_T = 0$ and $b_L = c_L = 0.01$, where the boson and fermion gases are in-phase. We note that this condition is consistent with the actual experiments by Kyoto group.

In Fig. 3, we show the time-dependence of $\Delta x_{L,T}$ for ^{170}Yb - ^{171}Yb , ^{170}Yb - ^{173}Yb and ^{174}Yb - ^{173}Yb mixtures. The $\Delta x_L(B)$ is found to oscillate monotonously (dotted lines), and show damping only in ^{174}Yb - ^{173}Yb mixture (Fig. 3e, dotted line) where the BF interaction is strongly repulsive. The boson $\Delta x_T(B)$ seems to be a superposition of two modes with different periods (solid lines); the contribution from the longer-period mode is rather dominant. The BTB mode seems to contribute very small except in ^{170}Yb - ^{173}Yb .

The $\Delta x_L(F)$ of the ^{170}Yb - ^{171}Yb and ^{174}Yb - ^{173}Yb mixtures show qualitatively similar behaviors: slight beat and damp (dotted lines). In the ^{170}Yb - ^{173}Yb mixture, the $\Delta x_L(F)$ (Fig. 3d, dotted line) is smaller than those in the other two mixtures, and comparable with $\Delta x_T(F)$ (Fig. 3d, solid line). In this mixture, we see the monotonous increase of the $\Delta x_T(F)$, which shows the expansion of the fermion gas in time.

In Figs. 4 (^{170}Yb - ^{171}Yb and ^{170}Yb - ^{173}Yb) and 5 (^{174}Yb - ^{173}Yb), we show the strength functions for the oscillation. We find that the $S_L(B)$ and $S_T(B)$ have one large peak at the same frequency in all mixtures; The ratios of the peak heights are $-S_L(B)/S_T(B) \approx 4.4$ (^{170}Yb - ^{171}Yb), 5.3 (^{170}Yb - ^{173}Yb), 4.2 (^{174}Yb - ^{173}Yb), which are close to the ratio in the BAB mode obtained in the scaling method, $-S_L(B)/S_T(B) = 4$ in Eq. (54).

In the ^{170}Yb - ^{171}Yb mixture, the $S_L(F)$ (Figs. 4a, solid line) shows one peak at $\omega \approx 0.26$, which is close to a peak of the $S_L(B)$ (dotted line), while the $S_T(F)$ has a peak around $\omega \approx 1.9$

(Figs. 4b). The ^{174}Yb - ^{173}Yb mixture shows the similar behavior except the $S_T(F)$, which shows a large peak at $\omega \approx 0.26$. The existences of large peaks in $S_L(F)$ result in the monotonous behaviors of the fermion oscillations in Fig. 3.

In the ^{174}Yb - ^{173}Yb mixture (Fig. 5), they show that the BTB and FTB modes exist in the oscillation, and that they are even-functions of τ . As explained in Ref. [34], these BTB and FTB oscillation modes are excited by the fermion-gas expansion seen in Fig. 3, though this expansion is very slow and small.

D. Transverse Breathing Oscillations

In this subsection, we discuss the oscillations caused by the transversally-deformed initial conditions, $b_T = c_T = 0.1$ and $b_L = c_L = 0$ (BF-in-phase condition), and $b_T = -c_T = 0.1$ and $b_L = c_L = 0$ (BF-out-of-phase condition).

In Fig. 6, we show the time-dependence of $\Delta x_{L,T}$ for the BF-in-phase initial condition, and we find monotonous oscillation in all mixtures. The amplitude modulation of Δx_L is seen in ^{170}Yb - ^{173}Yb (solid lines, Fig. 6c,d) in contrast with ^{170}Yb - ^{171}Yb and ^{174}Yb - ^{173}Yb ; it shows that a longitudinal mode with the longer period has been excited and superposed in the ^{170}Yb - ^{173}Yb mixture. The $\Delta x_{L,T}$ for the BF-out-of-phase initial condition are shown in Fig. 7; the oscillations of $\Delta x_{L,T}$ are not monotonous and show beats and dampings. In addition, the superposition of the longitudinal modes are seen even in ^{170}Yb - ^{171}Yb and ^{174}Yb - ^{173}Yb (dotted lines), especially, in the boson oscillations.

In transverse breathing oscillations, we find that the relative phase between the boson and fermion oscillations finally become in-phase independently of the initial conditions. It is confirmed in Fig. 8, where we plot the Δx_T in later time $115 < \tau < 145$. We will discuss this phenomena in the next subsection.

In Fig. 9 and 10, we show the strength functions $S_{L,T}$ of the oscillations in the ^{170}Yb - ^{171}Yb and ^{170}Yb - ^{173}Yb mixtures for the BF-in-phase (left panels) and BF-out-of-phase (right panels) conditions. The strength functions in ^{174}Yb - ^{173}Yb are very similar to those in ^{170}Yb - ^{171}Yb , and omitted here. For ^{170}Yb - ^{173}Yb , we plot both S_L ($1.1 < \omega$) and C_L ($\omega < 0.9$) in Fig. 10.

In all mixtures, the $S_T(B)$ has one sharp peak for both in-phase and out-of-phase conditions; in contrast, the $S_T(F)$ has one sharp peak for the BF-in-phase condition but one sharp-positive and broad-negative peaks for the BF-out-of-phase conditions (panels c,d in Figs. 9 and 10). The broad peaks for the BF out-of-phase condition correspond to the intrinsic FTB modes, but, for the in-phase initial conditions, the intrinsic FTB modes do not appear; the boson-gas oscillations

seem to influence on the fermion transverse breathing oscillation.

In the ^{170}Yb - ^{173}Yb mixture, the $C_L(B)$ has one sharp peak for both BF-in-phase and BF-out-of-phase conditions, and the S_L have very small peaks at the frequency of the peak position of $S_T(B)$. The long-period time-even modes in $C_L(B)$ are considered to be excited through the intrinsic BAB modes because they include both the longitudinal and transverse oscillation components (54). Thus, we find that the longitudinal and transversal oscillations are approximately time-even and time-odd in ^{170}Yb - ^{173}Yb . It explains that the longitudinal oscillation is not excited by the initial condition, and that the oscillation in ^{170}Yb - ^{173}Yb cannot be described with the superposition of the intrinsic oscillation.

In order to examine the above-mentioned results, we perform the calculation with the initial condition: $b_T = 0.1$ and $c_T = b_L = c_L = 0$, where only the boson gas is boosted along the transverse direction (Fig. 11). In the first one period, the relative phase between the boson and fermion oscillations are in-phase in ^{170}Yb - ^{173}Yb (repulsive BF interaction) and out-of-phase in ^{170}Yb - ^{171}Yb and ^{174}Yb - ^{173}Yb (attractive BF interaction); it is consistent with the spherical-trap calculations [33, 34]. The period of the boson oscillation is almost constant while that of the fermion oscillation varies in time; finally, the periods of them come to be equal and the relative phase of the boson and fermion oscillations become in-phase. This behavior is the same one with that in Fig. 8.

Now, we give a short explanation for the convergence to the BF in-phase oscillation in the transverse breathing oscillations independently of the initial condition (Figs. 8 and 11). As shown in Sec.II C, the ground-state boson density $\rho_B^{(g)}$ is a function of $\sqrt{\mathbf{r}_T^2 + \kappa_L^2 z^2}$ in the TF approximation; then, the longitudinal component of the effective fermion potential U_F in Eq. (9) becomes $\hat{z} \cdot \nabla_r U_F \approx \kappa_L \hat{r}_T \cdot \nabla_r U_F$. It shows that, when $\kappa_L \ll 1$, the force for the fermions along the longitudinal direction should be weak in comparison with that along the transverse direction; the intrinsic longitudinal motion of the fermion gas is very slow, and should be damped easily in early stage of oscillation. Then, it remains the forced oscillations by the boson gas with the longer period, and relative phase between the boson and fermion gases becomes in-phase.

E. Comparison with the Scaling Method

In this subsection, we discuss the BF-interaction dependence of the intrinsic-mode frequencies obtained in the TDGP+Vlasov approach, and compare them with those in the scaling method. Fig. 12 shows the \hbar_{BF}/g_{BB} -dependence of the boson and fermion intrinsic frequencies, which are obtained from the peak frequencies of the strength functions. The boson and longitudinal

fermion oscillations are found to have small h_{BF}/g_{BB} -dependence and the scaling method well reproduces the time-dependent calculations for these oscillations.

The fermion transverse oscillation has large interaction dependence, and the scaling method cannot reproduce the TDGP+Vlasov calculation (Fig. 12), especially at $h_{BF}/g_{BB} = 1.325$ (large repulsive BF interaction). In the scaling method calculation, the frequency of the FTB mode has the minimum around $h_{BF}/g_{BB} \sim 1$, and increases with increase of h_{BF}/g_{BB} in the region of $h_{BF}/g_{BB} > 1$ (Fig. 12, solid line); it comes from the structure change of the fermion ground-state density distribution into the surface-peaked shape (Fig. 2). On the other hand, in the TDGP+Vlasov approach, the density distribution and the velocity field varies in time in large amplitude oscillations. In the mixture of $h_{BF}/g_{BB} > 1$, the minimum position of the fermion potential is on the spherical surface at the border of the boson density distribution, and the fermion gas oscillates around this surface; it results in the frequency of the fermion transverse mode smaller than that in the sum-rule. The same discrepancy appears in the dipole and quadrupole oscillations in the spherical trap[31–34].

The longitudinal and transverse modes are found to decouple, and the frequencies of the boson and fermion intrinsic oscillations are very close. Thus the boson and fermion gases can easily oscillate with the same period; it also explains the convergence into the BF in-phase mode in the transverse oscillation, which has been discussed in the previous subsection.

V. SUMMARY

In this paper, we investigated the collective breathing oscillation of the BF mixtures of Yb isotopes in the deformed trap ($\kappa_L = 1/6$). Three combinations were examined: ^{170}Yb - ^{171}Yb , ^{170}Yb - ^{173}Yb and ^{174}Yb - ^{173}Yb , where the BF interactions are weakly repulsive, strongly attractive and strongly repulsive.

In the case of the weak BF interactions, the longitudinal and transversal breathing oscillations decouple in the largely deformed trap of prolate shape. Because of the large interaction energy coming from the condensed bosons, the boson breathing oscillations couple into the intrinsic BTB and BAB modes, and the intrinsic FTB and FAB modes are made from the fermion oscillations. The actual time-development processes are described by the superposition of these intrinsic modes.

The intrinsic transverse modes (BTB and FTB) have very close frequencies, and it is similar in the longitudinal modes (BAB & FLB); The frequencies of the two pairs separate very well.

In the spherical trap, the fermion oscillation have been shown to have the boson-forced oscillation

modes and the two intrinsic modes which correspond to the inside- and outside-fermion oscillations for the boson-distributed regions. In the present prolate deformed trap, no outside-fermion modes appear.

The longitudinal oscillations are rather simple; they are approximately monotonous in the boson and fermion oscillations in the all BF mixtures. For the in-phase initial condition, the BAB mode and the fermion intrinsic modes appear in the boson and fermion oscillations, respectively.

The transverse breathing oscillations have complex structures as clarified in IV D, and they depend on the strength of the BF interactions. In the fermion transverse breathing oscillations, we found that the fermion-gas frequencies vary in time and converges into the same frequencies with the boson oscillations and the relative phases of the boson and fermion oscillations become in-phase. This phenomenon are shown to be explained by the damping of the intrinsic fermion modes and the boson-forced oscillations.

In this paper, we discussed the collective oscillations of the BF mixtures of Yb isotopes at $T = 0$. In the actual experiments, which should be done at very low but $T \neq 0$ temperatures, thermal bosons should give some contributions; the introduction of such effects into the present approach through two-body collision terms [48, 49] should be done also in the future.

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- [1] E.A. Cornell and C.E. Wieman, Rev. Mod. Phys. **74**, 875 (2002);
W. Ketterle, Rev. Mod. Phys. **74**, 1131 (2002).
 - [2] F. Dalfovo, et al., Rev. Mod. Phys. **71**, 463 (1999).
 - [3] C.J. Pethik and H. Smith, "Bose-Einstein Condensation in Dilute Gases", Cambridge University Press (2002).
 - [4] J.O. Andersen, Rev. Mod. Phys. **76**, 599 (2004).
 - [5] A.G. Truscott, K.E. Streker, W.I. McAlexander, G.B. Partridge and R.G. Hulet, Science **291**, 2570 (2001).
 - [6] G. Roati, F. Riboli, G. Modugno and M. Inguscio, Phys. Rev. Lett. **89**, 150403 (2002).
 - [7] B. DeMarco and D.S. Jin, Science **285**, 1703 (1999);
S.R. Granade, M.E. Gehm, K.M. OHara, J.E. Thomas, Phys. Rev. Lett. **88**, 120405 (2002).
 - [8] F. Schreck, et al., Phys. Rev. Lett. **87**, 080403 (2001).
 - [9] Z. Hadzibabic, et al., Phys. Rev. Lett. **88**, 160401 (2002); *ibid* **91**, 160401 (2003).

- [10] M. Modugno, F. Ferlaino, F. Riboli, G. Roati, G. Modugno, M. Inguscio, *Science* **297**, 2240 (2002);
Phys. Rev. A **68**, 043626 (2003).
- [11] H. Feshbach, *Ann. Phys. (NY)* **19**, 287 (1962).
- [12] K. Mølmer, *Phys. Rev. Lett.* **80**, 1804 (1998).
- [13] M. Amoruso, A. Minguzzi, S. Stringari, M. P. Tosi and L. Vichi, *Eur. Phys. J. D* **4**, 261 (1998).
- [14] T. Miyakawa, K. Oda, T. Suzuki and H. Yabu, *J. Phys. Soc. Japan* **69**, 2779 (2000).
- [15] M.J. Bijlsma, B.A. Heringa and H.T.C. Stoof, *Phys. Rev. A* **61**, 053601 (2000).
- [16] L. Vichi, M. Inguscio, S. Stringari, and G.M. Tino, *Eur. Phys. J. D* **11**, 335 (2000).
- [17] L. Vichi, M. Inguscio, S. Stringari and G.M. Tino, *J. Phys. B: At. Mol. Opt. Phys.* **31** L899 (1998).
- [18] N. Nygaard and K. Mølmer, *Phys. Rev. A* **59**, 2974 (1999).
- [19] X.X. Yi and C.P. Sun, *Phys. Rev. A* **64**, 043608 (2001).
- [20] L. Viverit, C.J. Pethick and H. Smith, *Phys. Rev. A* **61**, 053605 (2000).
- [21] P. Capuzzi and E.S. Hernández, *Phys. Rev. A* **66**, 035602 (2002).
- [22] P. Capuzzi, A. Minguzzi and M.P. Tosi, *Phys. Rev. A* **68**, 033605 (2003).
- [23] T. Miyakawa, T. Suzuki and H. Yabu, *Phys. Rev. A* **64**, 033611 (2001).
- [24] R. Roth and H. Feldmeier, *Phys. Rev. A* **65**, 021603(R) (2002).
- [25] P. Capuzzi and E. S. Hernández, *Phys. Rev. A* **64**, 043607 (2001).
- [26] T. Sogo, T. Miyakawa, T. Suzuki and H. Yabu, *Phys. Rev. A* **66**, 013618 (2002);
T. Sogo, T. Suzuki and H. Yabu, *Phys. Rev. A* **68**, 063607 (2003).
- [27] T. Miyakawa, K. Oda, T. Suzuki and H. Yabu, *J. Phys. Soc. Jpn* **69**, 2779 (2000).
- [28] T. Miyakawa, T. Suzuki and H. Yabu, *Phys. Rev. A* **62**, 063613 (2000).
- [29] A. Minguzzi and M.P. Tosi, *Phys. Lett. A* **268**, 142 (2000).
- [30] S.K. Yip, *Phys. Rev. A* **64**, 023609 (2001).
- [31] T. Maruyama, H. Yabu and T. Suzuki, *Phys. Rev. A* **72**, 013609 (2005).
- [32] T. Maruyama, H. Yabu and T. Suzuki, *Laser Physics*, **15**, 656 (2005).
- [33] T. Maruyama and G.F. Bertsch, *Phys. Rev. A* **77**, 063611 (2008).
- [34] T. Maruyama and H. Yabu, *Phys. Rev. A* **80**, 043615 (2009).
- [35] Y. Takasu, et al., *Phys. Rev. Lett.* **91**, 040404 (2003).
- [36] T. Fukuhara, S. Sugawa, and Y. Takahashi, *Phys. Rev. A* **76**, 051604(R) (2007).
- [37] T. Fukuhara, Y. Takasu, M. Kumakura and Y. Takahashi, *Phys. Rev. Lett.* **98**, 030401 (2007).
- [38] K. Enomoto, M. Kitagawa, K. Kasa, S. Tojo and Y. Takahashi, *Phys. Rev. Lett.* **98**, 203201 (2007);
M. Kitagawa, et.al., *Phys. Rev. A* **77**, 012719 (2008).
- [39] G.F. Bertsch, *Nucl. Phys. A* **249**, 253 (1975);
D.M. Brink and Leobardi, *Nucl. Phys. A* **258**, 285 (1976).
- [40] G.F. Bertsch and K. Stricker, *Phys. Rev. C* **13**, 1312 (1976);
T. Suzuki, *Prog. Theor. Phys.* **64**, 1627 (1980).
- [41] T. Maruyama and G.F. Bertsch, *Phys. Rev. A* **73**, 013610 (2006).
- [42] T. Maruyama and T. Nishimura, *Phys. Rev. A* **75**, 033611 (2007).

- [43] L.P. Kadanoff and G. Baym, “Quantum Statistical Mechanics” (1962), NewYork.
- [44] L. Vichi and S. Stringari, Phys. Rev. **A60**, 4734 (1999).
- [45] O. Bohigas, A.M. Lane and J. Martorell, Phys. Rep. **51**, 267 (1979),
- [46] H. Hu, A. Minguzzi, X.J. Liu and M.P. Tosi, Phys. Rev. Lett., **93**, 190403 (2004).
- [47] R.W. Hockney AND J.W. Eastwood, ”Computer simulations using particles”, McGraw-Hill, New York, 1981;
C.Y. Wong, Phys. Rev. **C25**, 1460 (1982).
- [48] B. Jackson and E. Zaremba, Phys. Rev. **A66**, 033606 (2002).
- [49] G.F. Bertsch and S. Das Gupta, Phys. Rep. **160** (1988) 189.

System	g_{BB}	h_{BF}	h_{BF}/g_{BB}
$^{170}\text{Yb}-^{171}\text{Yb}$	5.598×10^{-2}	3.207×10^{-2}	0.573
$^{170}\text{Yb}-^{173}\text{Yb}$	5.598×10^{-2}	-7.126×10^{-2}	-1.273
$^{174}\text{Yb}-^{173}\text{Yb}$	8.902×10^{-2}	11.799×10^{-2}	1.325

TABLE I: The coupling constants g_{BB} and h_{BF} of the boson-boson and the boson-fermion interactions in $^{170}\text{Yb}-^{171}\text{Yb}$, $^{170}\text{Yb}-^{173}\text{Yb}$ and $^{174}\text{Yb}-^{173}\text{Yb}$.

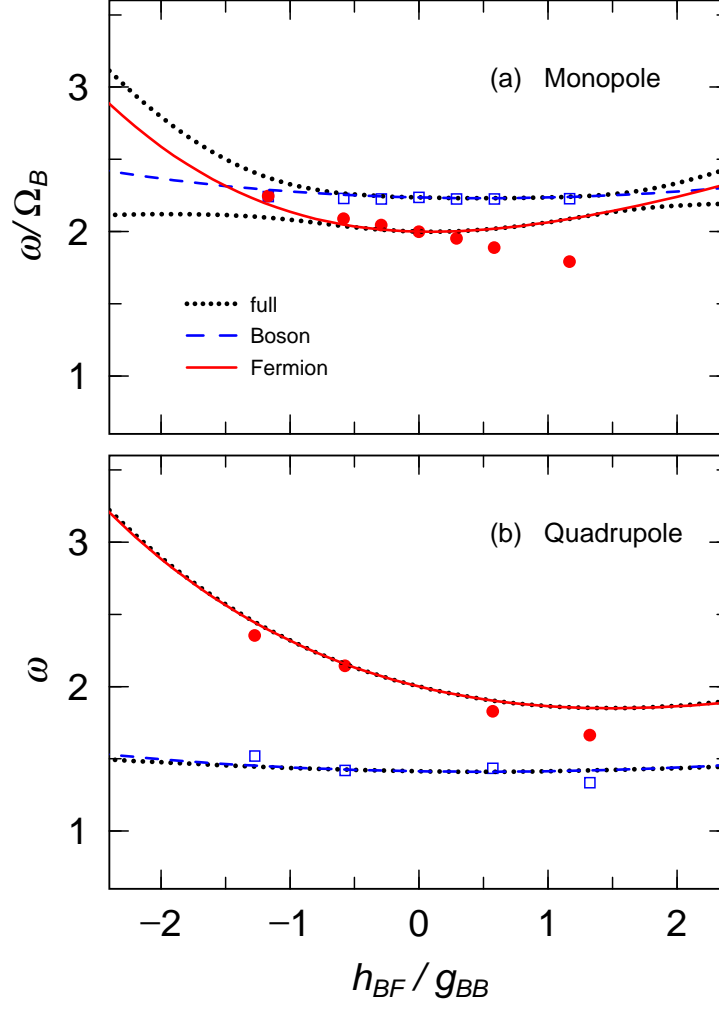


FIG. 1: (Color online) The frequencies of the monopole (a) and the quadrupole oscillations (b) in the spherically-symmetric system ($\kappa_L = 1$) in the scaling method: the co-oscillating (dotted lines), the boson-intrinsic (dotted lines), and the fermion-intrinsic (solid lines) modes. The open squares and full circles denote the numerical results obtained in the TDGP+Vlasov calculation for the boson- and fermion-intrinsic frequencies, which are taken from Refs. [31] (monopole) and [34] (quadrupole).

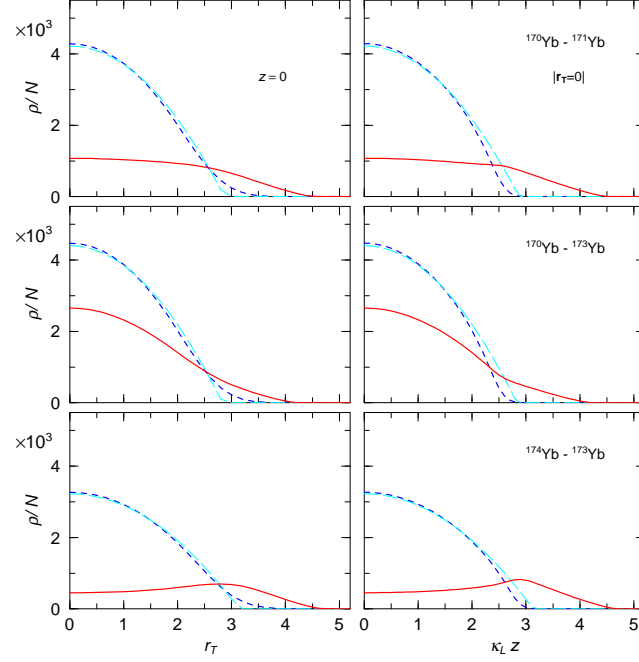


FIG. 2: (Color online) The ground-state density distributions of the BF mixtures ^{170}Yb – ^{171}Yb (upper panels), ^{170}Yb – ^{173}Yb (middle panels) and ^{174}Yb – ^{173}Yb (bottom panels) at $z = 0$ (transverse direction, left panels) and at $r_T = 0$ (longitudinal direction, right panels). Dotted and solid lines are for the bosons and fermions, respectively. The thin long dotted lines indicate the boson density in the TF approximation.

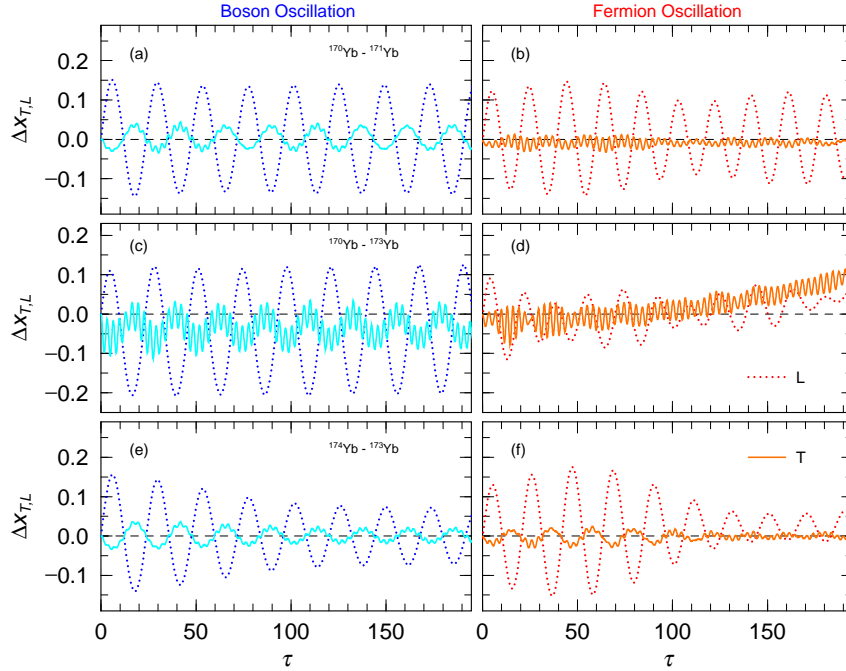


FIG. 3: (Color online) Time evolution of $\Delta x_{L,T}(B)$ (boson oscillation, left figures) and $\Delta x_{L,T}(F)$ (fermion oscillation, right figures) for the in-phase initial condition: $b_T = c_T = 0$ and $b_L = c_L = 0.01$. The upper, middle, and lower panels are for the ^{170}Yb – ^{171}Yb , ^{170}Yb – ^{173}Yb and ^{174}Yb – ^{173}Yb mixtures. The dotted and solid lines represent Δx_L and Δx_T , respectively.

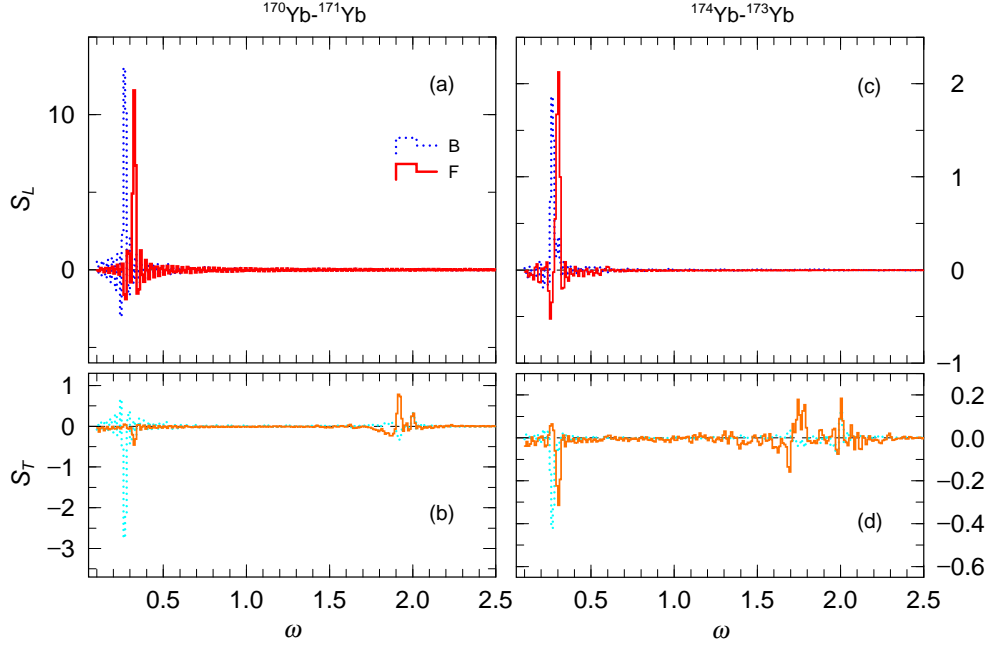


FIG. 4: (Color online) The strength functions S_L (longitudinal, upper figures) and S_T (transversal, lower figures) of the boson and fermion oscillations for the in-phase initial condition. The left and right panels are for the ^{170}Yb - ^{171}Yb and ^{174}Yb - ^{173}Yb mixtures.

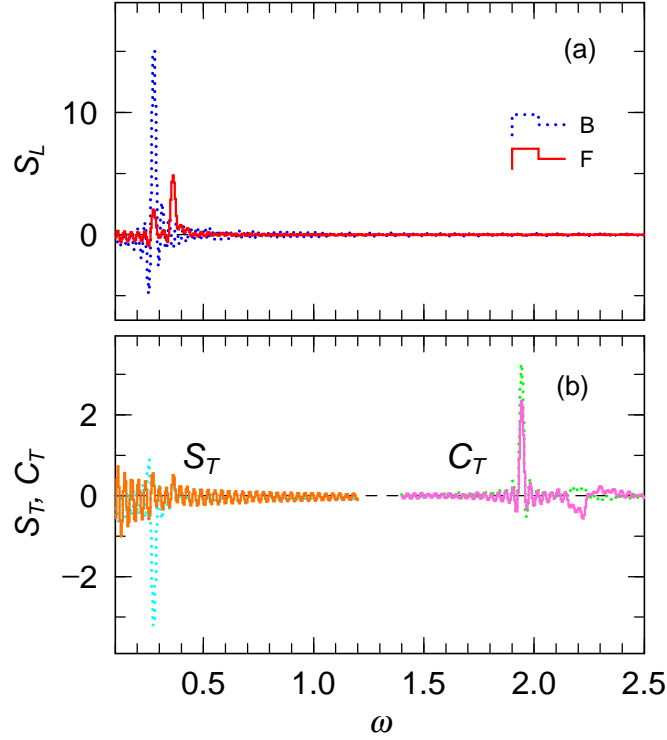


FIG. 5: (Color online) The strength functions S_L (longitudinal, upper figure) and S_T & C_T (transversal, lower figure) of the boson (dotted lines) and fermion (solid lines) oscillations in the ^{170}Yb - ^{173}Yb mixture for the in-phase initial condition. In the lower panel, S_T and C_T are plotted in $\omega < 1.3$ and $1.4 < \omega$, respectively.

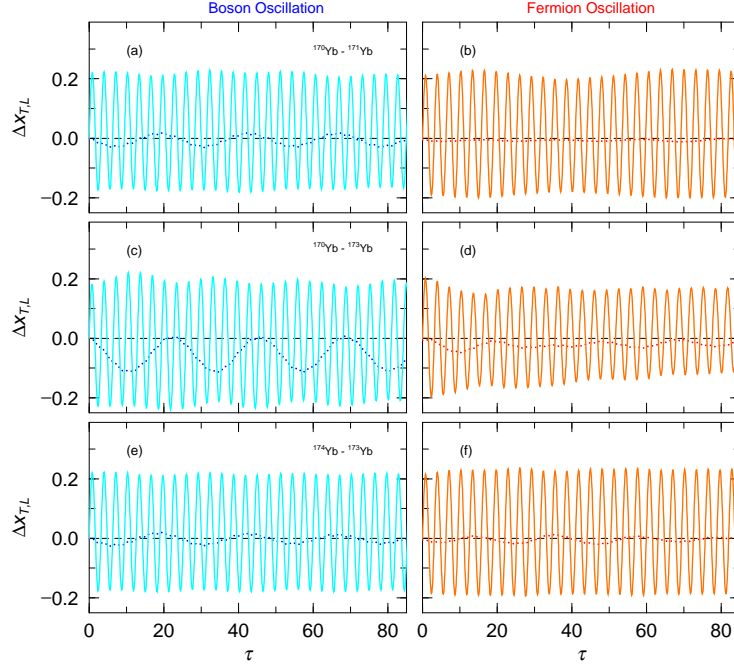


FIG. 6: (Color online) Time evolution of $\Delta x_{L,T}(B)$ (boson oscillation, left figures) and $\Delta x_{L,T}(F)$ (fermion oscillation, right figures) for the in-phase initial condition: $b_T = c_T = 0.1$ and $b_L = c_L = 0$. The upper, middle, and lower panels are for the $^{170}\text{Yb}-^{171}\text{Yb}$, $^{170}\text{Yb}-^{173}\text{Yb}$ and $^{174}\text{Yb}-^{173}\text{Yb}$ mixtures. Dotted and solid lines represent Δx_L and Δx_T , respectively.

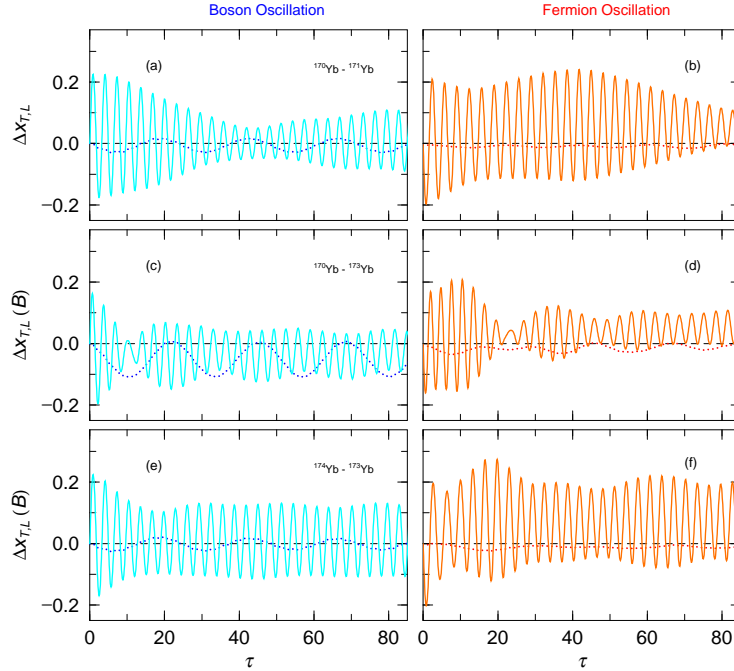


FIG. 7: (Color online) Time evolution of $\Delta x_{L,T}(B)$ (boson oscillation, left figures) and $\Delta x_{L,T}(F)$ (fermion oscillation, right figures) for the out-of-phase initial condition: $b_T = -c_T = 0.1$ and $b_L = c_L = 0$. The upper, middle, and lower panels are for the $^{170}\text{Yb}-^{171}\text{Yb}$, $^{170}\text{Yb}-^{173}\text{Yb}$ and $^{174}\text{Yb}-^{173}\text{Yb}$ mixtures. Dotted and solid lines represent Δx_L and Δx_T , respectively.

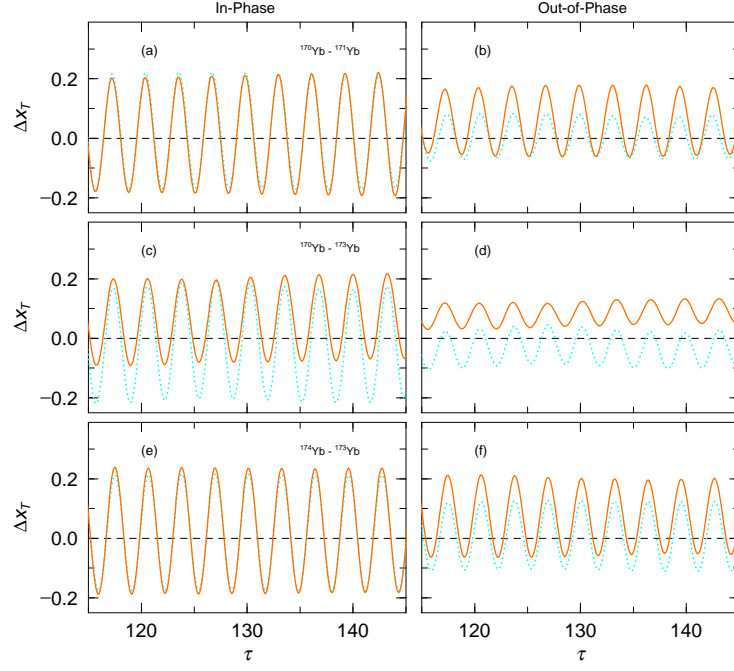


FIG. 8: (Color online) The time evolutions of Δx_T in later time $115 < \tau < 145$, for the BF-in-phase (left panels) and out-of-phase (right panels) initial conditions in the $^{170}\text{Yb} - ^{171}\text{Yb}$ (upper panels), $^{170}\text{Yb} - ^{171}\text{Yb}$ (middle panels) and $^{170}\text{Yb} - ^{171}\text{Yb}$ (lower panels) mixtures. Dotted and solid lines represent the results of the boson and fermion oscillations.

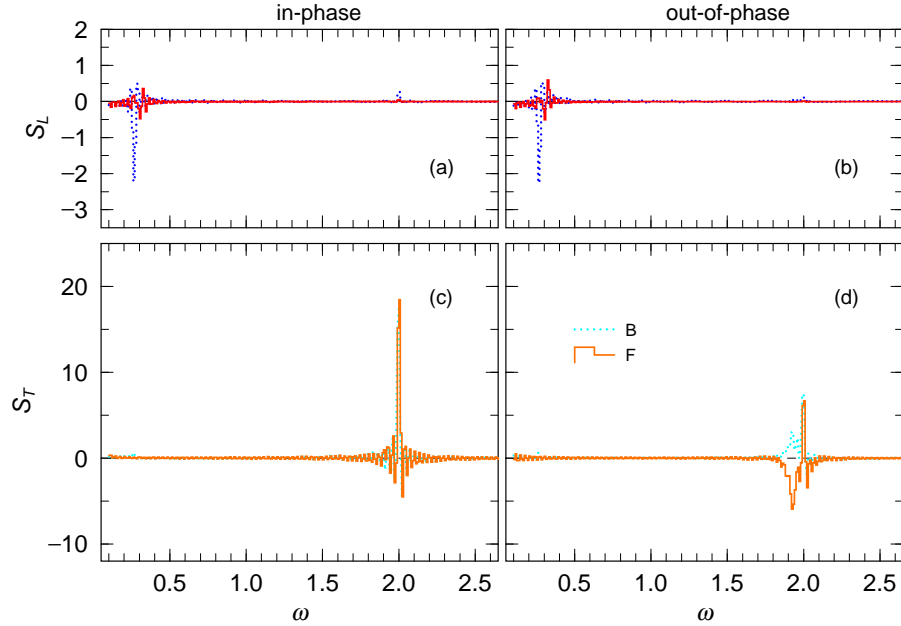


FIG. 9: (Color online) The strength functions S_L (longitudinal, upper figures) and S_T (transversal, lower figures) of the boson and fermion oscillations in $^{170}\text{Yb} - ^{171}\text{Yb}$ for the transversally-deformed initial condition: the left and right panels are for the BF-in-phase and BF-out-of-phase conditions. Dotted and solid lines represent the results of the boson and fermion oscillations, respectively.

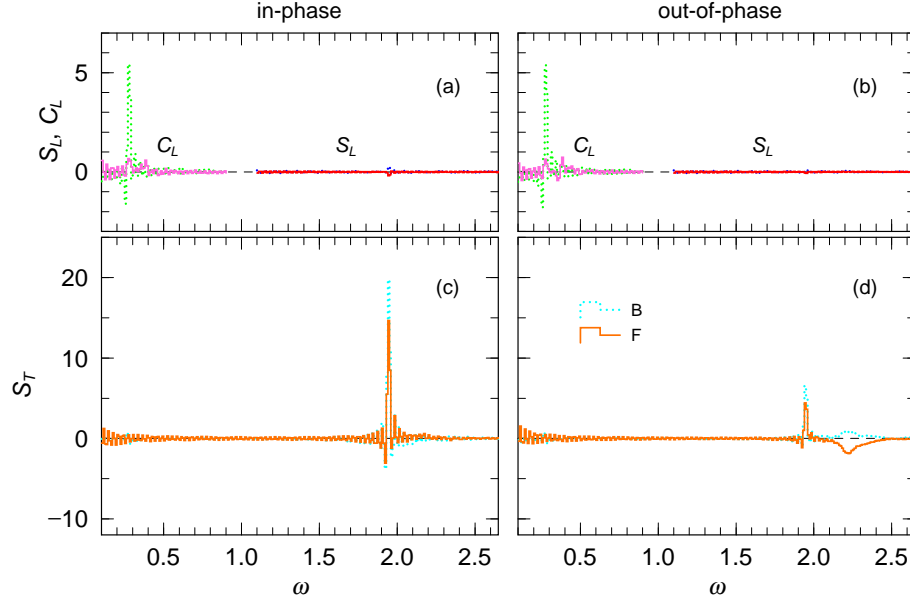


FIG. 10: (Color online) The strength functions S_L (longitudinal, upper figures) and S_T (transversal, lower figures) of the boson and fermion oscillations in ^{170}Yb – ^{173}Yb for the transversally-deformed initial condition: the left and right panels are for the BF-in-phase and BF-out-of-phase conditions. Dotted and solid lines represent the results of the boson and fermion oscillations, respectively.

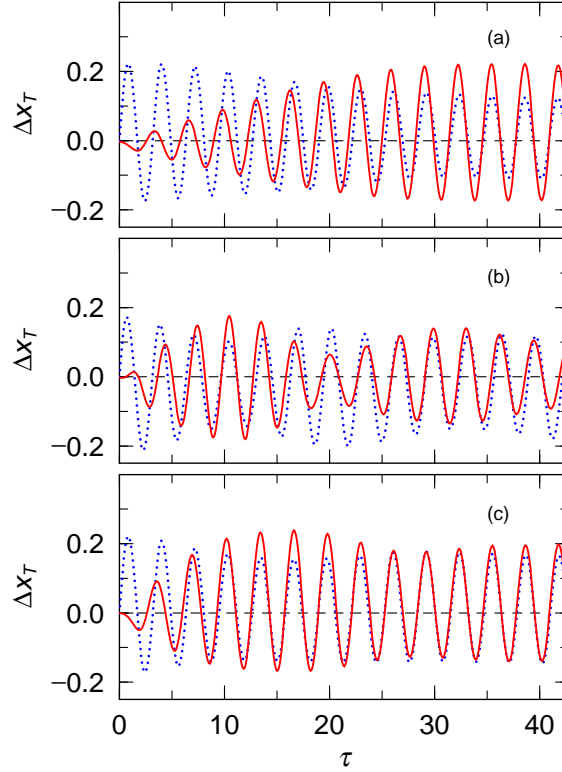


FIG. 11: (Color online) Time evolution of Δx_T in the ^{170}Yb – ^{171}Yb (a), ^{170}Yb – ^{173}Yb (b) and ^{174}Yb – ^{173}Yb (c) mixtures for the initial condition: $b_T = 0.1$ and $c_T = b_L = c_L = 0.01$. Solid and dotted lines are for the bosons and fermions, respectively.

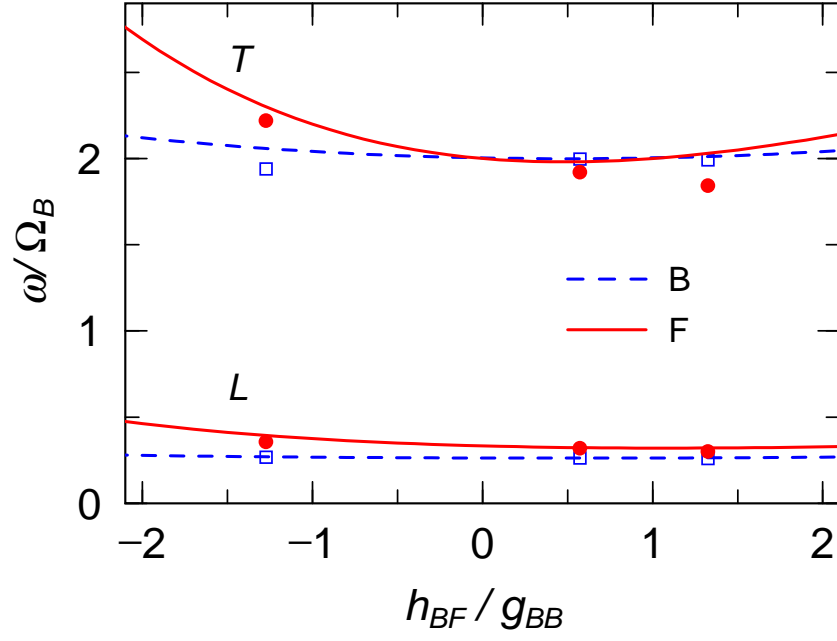


FIG. 12: (Color online) Intrinsic frequencies of the breathing oscillations of the boson and fermion gases obtained in the TDGP+Vlasov approach; circles and squares are for the longitudinal and transverse oscillation modes in the ^{170}Yb – ^{171}Yb , ^{170}Yb – ^{173}Yb and ^{174}Yb – ^{173}Yb mixtures. Dotted and solid lines represent the results of the longitudinal and transverse modes in the scaling method.